MHD Turbulence II

- → Anisotropic Cascades and Critical Balance ↔ A Closer Look.
- → Extending the 4/5 Law.
- → Selective Decay and Relaxation.
- → 2D MHD — A Study in Turbulent Relaxation.

i.) Anisotropic Cascades and Critical Balance — A Closer Look.

Recall: I-K Phenomenology:

$$\epsilon \cong \frac{Z(\ell)^2}{\tau_{Tr}(\ell)} \qquad (B_0 \text{ weak, but } B_{rms} \text{ generated})$$

$$1/\tau_{Tr}(\ell) \cong \frac{Z(\ell)^2}{\ell^2}\tau_A \qquad \tau_A \sim \frac{\ell}{b_0}$$

$b_0 \rightarrow rms$

$$\Rightarrow Z_\ell \sim (b_0\epsilon)^{1/4}\,\ell^{1/4}$$

$$E(k) \sim \sqrt{\epsilon b_0}\; k^{-3/2}$$

and with strong $\underline{B_0}$:

$$\epsilon \sim \frac{Z(\ell_\perp)^2\, Z(\ell_\perp^2)}{\ell_\perp^2\, |k_\parallel V_A|}$$

so W.T.T. Alfvenic cascade:

$$E(k_\perp, k_\parallel) \sim (\epsilon V_A)^{1/2} / k_\parallel^{1/2} k_\perp^2$$

~ "hard" in $k_\perp$!

However, note:

$$Z(\ell_\perp) \sim \delta B(\ell_\perp) \sim (\epsilon / k_\parallel V_A /)^{1/4} \ell_\perp^{1/2}$$

so

$$\delta B \cdot \nabla_\perp \sim \frac{1}{\ell_\perp^{1/2}} (\epsilon / k_\parallel V_A /)^{1/4}$$

But recall:

— Alfven wave:

$$\omega \simeq k_\parallel V_A$$

derived from:

$$\partial_t A = B_0 \nabla_\parallel \phi + \ldots$$

$$\partial_t \nabla^2 \phi = B_0 \nabla_\parallel J_\parallel + \ldots$$

$$\nabla_\parallel = \underset{\text{Linear}}{\partial_z} + \underset{\text{Nonlinear}}{\frac{\delta B_\perp}{B_0} \cdot \nabla_\perp}$$

Ratio $\dfrac{\text{Nonlinear}}{\text{Linear}} = Ku = \dfrac{\frac{\delta B_\perp}{B_0}\cdot \nabla_\perp}{\partial z}$

$\downarrow$

Kubo #

$\Rightarrow$

$$Ku \sim \frac{\delta B\, l_{ac}}{B_0\, \Delta_\perp}$$

$l_{ac} \rightarrow$ parallel autocorrelation length

$\Delta_\perp \rightarrow \perp$ correlation length.

see stochastic fields discussion of Phys 235 2016.

Point:

$$B_0 \partial_z C + \delta B_\perp \cdot \nabla_\perp C = 0$$

$$\partial_z C + \frac{\delta B_\perp}{B_0} \cdot \nabla_\perp C = 0$$

$Ku < 1$ $\rightarrow$ C evolves by many kicks in $\Delta_\perp$

$\rightarrow$ diffusion

$\rightarrow$ in WTT wave interactions are diffusive in character.

$Ku > 1$ $\rightarrow$ C scattered $> \Delta_\perp$ in one step

$\rightarrow$ for transport in random media $\rightarrow$ percolation

Analogy $\quad \partial_t c + \underline{v} \cdot \nabla c = 0$

$$Ku = \frac{\tilde{v}\, \tau_{ac}}{\Delta_\perp}$$

So we have a concern:

→ physics of MHD turbulence understood in terms of Alfven wave interactions.

→ but scalings of WTT spectrum suggest that wave character lost as cascade progresses

i.e.

$$Ku \sim \frac{k_\parallel^{-1}}{\ell_\perp^{1/2}} \left[\epsilon \, |k_\parallel V_A| \right]^{1/4}$$

$$\uparrow \quad \text{as} \quad \ell_\perp \downarrow$$

i.e. How high can Ku # go and still be consistent with physics of Alfven Wave Cascade

⇒ $\boxed{\text{Critical Balance Conjecture}}$

(GS '95, K+P '78)

$\Rightarrow$ $\boxed{\text{MHD inertial range in strong field will set at } ku \sim 1.}$

i.e. $\rightarrow$ $\delta B_\perp \cdot v_\perp \sim \frac{Z(\ell_\perp)}{\ell_\perp} \sim B_0 v_\parallel$

so $\frac{Z(\ell_\perp)}{\ell_\perp} \simeq k_\parallel V_A$

$\rightarrow \frac{\tau_A}{\tau_{eddy}} \rightarrow 1$ $\quad \tau_{Tr} \rightarrow \tau_{eddy} \sim \tau_A.$

i.e. all timescales equalize

$\rightarrow ku \sim 1$ is maximum ku and still retain Alfvenic character.

$\rightarrow$ why?

Recall:

- WTT $\quad \tau_{Tc} \sim \tau_{ac}$ wave $\quad$ {Triad coherence set by wave dispersion

$\rightarrow \pi \delta(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})$

- STT - Renormalized Theory

$\tau_{Tc} \sim \tau_c$ $\quad$ {Triad coherence set by nonlinear scattering, itself.

$\rightarrow 1/\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}$

$= \Theta_{\underline{k}, \underline{k}', \underline{k}''}$

So, renormalized wave interaction theory ⇒

$$\Theta_{\underline{k},\underline{k}',\underline{k}''} = \frac{\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''}}{(\omega_{\underline{k}} - \omega_{\underline{k}''} - \omega_{\underline{k}'})^2 + (\Delta\omega_{\underline{k}} + \Delta\omega_{\underline{k}'} + \Delta\omega_{\underline{k}''})^2}$$

→ recovers both limits ✓

Now, $\Theta_{\underline{k},\underline{k}',\underline{k}''}$ clearly sets T_{Tr}.

So, can re-write phenomenological transfer balance as:

$$\epsilon \sim \frac{1}{\ell_1^2} \frac{Z(\ell_1)^2 Z(\ell_1)^2}{\tilde{\tau}_c(\ell_1)}$$

$$1/\tau_c(\ell_1) = 1 \Big/ \left[\underset{①}{(k_{\shortparallel} V_A)^2} + \underset{②}{\left(\frac{Z(\ell_1)}{\ell_1} \right)^2} \right]^{1/2}$$

comparable at $k_{\shortparallel} \sim 1$

by analogy with $\Theta_{\underline{k},\underline{k}',\underline{k}''}$.

① > ② → W.T.T.

① ≤ ② → S.T.T.

$$\epsilon \sim \frac{1}{\ell_\perp} \frac{z(\ell_\perp)^2 z(\ell_\perp)^2}{z(\ell_\perp)/\ell_\perp}$$

$$\sim z(\ell_\perp)^3/\ell_\perp$$

and $\boxed{z(\ell_\perp) \sim (\epsilon \ell_\perp)^{1/3}}$ — Back to k 41 !

Point: $\langle z(k)^2 \rangle \sim \epsilon^{2/3} k_\perp^{-5/3}$ — G-S spectrum.
but different physics!

— softer than W.T.T.

— $\frac{z(\ell_\perp)}{\ell_\perp}$ vs. $k_\parallel V_A$

— "Great Power Lord in sky"

$$\frac{(\epsilon \ell_\perp)^{1/3}}{\ell_\perp} \sim \frac{\epsilon^{1/3}}{\ell_\perp^{2/3}}$$ → rate increases as $\ell_\perp \downarrow$

→ contrast constant $|k_\parallel V_A|$

— $$\frac{z(\ell_\perp)}{\ell_\perp} \sim \frac{\epsilon^{1/3}}{\ell_\perp^{2/3}} \to \frac{\delta B}{B_0} \cdot \nabla_\perp$$

then $k_{\parallel} \sim 1$ $\Rightarrow$

$$\frac{\epsilon^{1/3}}{\ell_{\perp}^{2/3}} \sim k_{\parallel}$$

$$\Rightarrow \boxed{k_{\parallel} \sim \epsilon^{1/3} k_{\perp}^{2/3}} \quad \text{— GS cone.}$$

→ - Critical Balance is a hypothesis

- Plausible answer to question of "how maintain Alfvenic cascade in state of strong (i.e. non-weak) turbulence?"

- anisotropy of spectrum supported by simulations (cf. Galtier).

BUT

- hypothesis, only

→ Computational support semi-quantitative.

→ 5/3 vs 3/2 etc. still ongoing.

$\Rightarrow$ A word about triads.

In wave turbulence cascade, must satisfy:

$$\underline{k} = \underline{p} + \underline{q}$$

$$\omega_{\underline{k}} = \omega_{\underline{p}} \pm \omega_{\underline{q}} \qquad \text{(WTT)}$$

$\rightarrow$ resonance criterion

Conditions satisfied by:

$q_{\parallel} = 0$ (i.e. $\underline{q}$ is a cell, driven by beats)

so $k_{\parallel} = p_{\parallel}$

$\underline{k_{\perp}} = \underline{p_{\perp}} + \underline{q_{\perp}}$

and $\omega_{k} = \omega_{p} \pm \cancel{\phi}$

$\Rightarrow$ – deformation of Alfvenic wave packet directly related to its interaction with 2D part of wave packet travelling in opposite direction.

– interaction passive w/r $k_{\parallel}$
$\Rightarrow$ $\perp$ transfer in long time limit.

ii.) 4/5 Law — See Lecture I.

iii.) Cascades and Relaxation

$\Rightarrow$ Selective Decay

Recall: Taylor Relaxation { 3D; 2D – "Taylor in Flatland" }

Argued: $\int d^3x\, B^2/8\pi$ minimized subject to constraint of $\int d^3x\, A \cdot B$ conserved.

$\Rightarrow \quad J_{\parallel} = J \cdot B / B^2 \rightarrow$ const.

(2D $J/A \rightarrow$ const.).

Arguements heuristic. { Power counting ($k_\perp$); stoch. fields; ⋮ }

Now, – dissipation at small scale η, ν

– expect energy transfer to small scale!

⋮

- Inverse {cascade / transfer} of magnetic helicity would set up "selective Decay" scenario

i.e. magnetic energy scattered to small scale and dissipated
$\Rightarrow$ relaxation

magnetic helicity inverse cascades
$\Rightarrow$ avoids dissipation. Constraint, as survives.

c.f. Frisch (75), Pouquet, et. al. (76)
(posted)
see also: Montgomery

- why, where from?

$\Rightarrow$ Primarily: Statistical Mechanics

$\Rightarrow$ c.f.: Frisch '75, though not transparent.

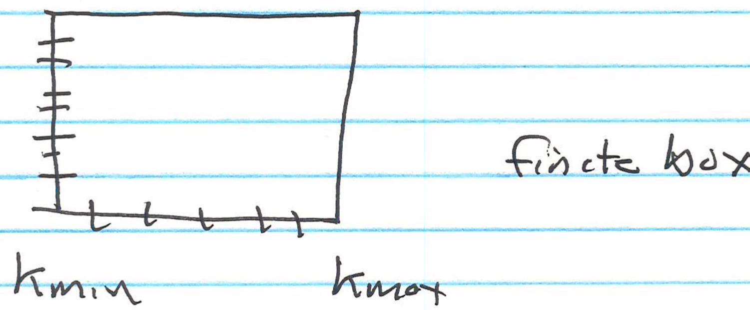
easier $\rightarrow$ "Taylor in Flatland" Problem.

Recall: Relaxation $\begin{cases} \text{minimizes } \langle B^2 \rangle \\ \text{conserving } \langle A^2 \rangle \end{cases}$.

Does this follow from Selective Decay?

$\Rightarrow$ Explore Absolute Equilibrium

i.e.



- remove forcing, dissipation etc.
- input excitation.

For 2D MHD (ignoring cross helicity):

have $A \rightarrow X_i$

$\hookrightarrow$ Modal amplitude

so

$$E_M = \sum_{i=1}^{N} k_i^2 X_i^2$$

$$H = \sum_{i=1}^{N} X_i^2 \qquad - \langle A^2 \rangle$$

$$\phi \to Y_i$$

$$E_K = \sum_{i=1}^{N} k_i^2 Y_i^2$$

Now,

$$\begin{matrix} H & \to & \alpha \\ E = E_{H} + E_K & \to & \beta \end{matrix} \Bigg\rangle \text{ conserved}$$

conserved, so pdf of this closed system is given by micro-canonical ensemble/distribution:

$$P(X, Y) = \underset{\substack{\downarrow \\ \text{norm}}}{c} \exp\left[-\sum_{i=1}^{N} \left[(\alpha + \beta k_i^2) X_i^2 + \beta k_i^2 Y_i^2 \right] \right]$$

and can integrate out Y_i (KE) part, so:

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2 \right]$$

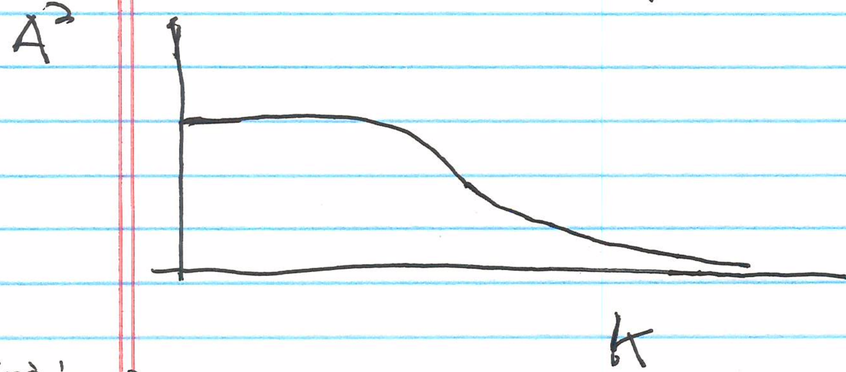
then:

$$\langle A^2(k) \rangle = \int dX_i \, X_i^2 \, P(X_i)$$

$$= 1/(\alpha + \beta k^2)$$

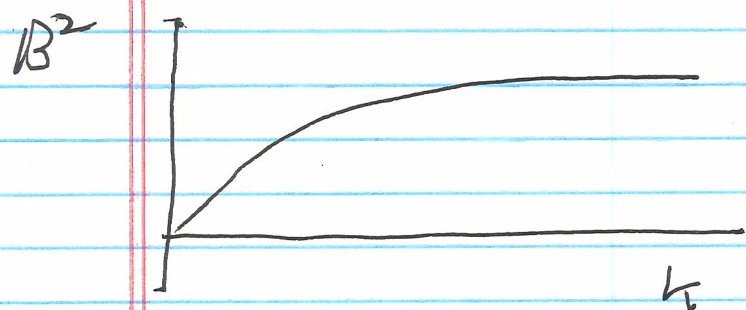
$$\langle B^2(k) \rangle = \left[k^2 / (\alpha + \beta k^2) \right]$$

∞ observe immediately:



"A^2 wants remain at large scale"

$k_{min} \to 0$



"B^2 approaches equipartition"

⇒ A^2 distribution most populated at larger scales. Decays at small

⇒ B^2 distribution most populated at smaller. Approaches equipartition at small scale.

∴ - suggests A^2 populates large scales, B^2 approaches equipartition.

- suggestive of inverse cascade of A^2, along with forward cascade of energy.

- supports Selective Decay Hypothesis as foundation for "Taylor in Flatland".

- similar story for Magnetic Helicity, though more laborious.

N.B. For 2D Fluid:

$$E = \int d^2x \, (\nabla\phi)^2 \qquad \text{- energy}$$

$$\Omega = \int d^2x \, (\nabla^2\phi)^2 \qquad \text{- enstrophy}$$

$$\Omega_i = k_i^2 E_i$$

$$\underline{v} \to \underline{X}_i$$

$\therefore$

$$P(X) = c \exp\left[-\sum_{i=1}^{N} (\alpha + \beta k_i^2) X_i^2\right]$$

so

$$E(k) = \langle v^2(k) \rangle = 1/(\alpha + \beta k^2)$$

$$\Omega(k) = k^2/(\alpha + \beta k^2)$$

similar suggestion of dual cascade and minimum enstrophy state.

→ Is this story true?

⇒ What does dynamics tell us?
∴ Consider interactions in 2D MHD.

Observe:

- Reduced MHD

$$\frac{\partial \psi}{\partial t} + \nabla_\perp \phi \times \hat{z} \cdot \nabla_\perp \psi = B_0 \partial_z \phi + \eta \nabla_\perp^2 \psi$$

- 2D MHD

$$\frac{\partial \psi}{\partial t} + \nabla_\perp \phi \times \hat{z} \cdot \nabla_\perp \psi = \eta \nabla_\perp^2 \psi$$

so; with strong $\underline{B_0}$:

$$\langle A \cdot B \rangle \rightarrow \langle \psi \rangle \underline{B_0}$$

so mean $\langle \psi \rangle$ in 2D captures magnetic helicity dynamics in strongly magnetized system.

For $\langle A^2 \rangle_k$ transfer, consider closure of $\partial_t \langle A^2 \rangle$ equation, much akin to wave kinetics, though closure required.

See: Diamond, Hughes, Kim (posted).

Can write (see DHK):

$$\frac{1}{2}\left[\partial_t \langle A^2\rangle_{\underline{k}} + T(k)\right] = -\Gamma_A(k)\frac{\partial\langle A\rangle}{\partial x} - \eta\langle B^2\rangle_{\underline{k}}$$

triplet

$$\langle \nabla\cdot\langle \underline{V}A^2\rangle\rangle_{\underline{k}}$$

flux

$$\Gamma_A = \left[\tilde{\tau}_c^{\phi}(\underline{k})\langle V^2\rangle_{\underline{k}} - \tilde{\tau}_c^{A}(\underline{k})\langle B^2\rangle_{\underline{k}}\right]$$

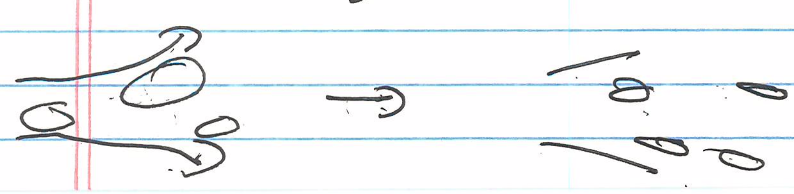
$$T_{\underline{k}} = \sum_{\underline{k}'}(\underline{k}\cdot\underline{k}'\times\hat{z})^2\,\Theta_{\substack{k,k'\\k''}}\left\{\overset{(1)}{\langle\phi^2\rangle_{\underline{k}'}} - \overset{(2)}{\frac{(k'^2-k^2)}{|\underline{k}+\underline{k}'|^2}}\langle A^2\rangle_{\underline{k}'}\right\}\{A^2_{\underline{k}}\}$$

$$\overset{(3)}{-\sum_{\substack{\underline{k}=\underline{p}+\underline{q}\\ \underline{p},\underline{q}}}}(\underline{p}\cdot\underline{q}\times\hat{z})^2\,\Theta_{\underline{k},\underline{p},\underline{q}}\langle A^2\rangle_{\underline{p}}\langle\phi^2\rangle_{\underline{q}}$$

(1), (3) → coherent damping, incoherent emission

→ akin to scattering of passive scalar, → small scale / chop-up.

→ conserve $\langle\psi^2\rangle$ upon $\sum_{k}$ together.



(2) → coherent damping/growth - from back reaction $(J \times B)$ into Ohm's Law:

→ reshuffle $\langle A^2 \rangle$ to larger scale. Sign k'^2 vs k^2!

→ $\sum_k$ conserves $\langle A^2 \rangle$ independently.

→ correspondence to condensation of wires (currents) attracting.

(1) + (2) → net effective resistivity sign.

→ see Γ_A, too. - 'negative resistivity'
- Alfvenized state

$\Rightarrow E_K > E_M \Rightarrow$ $\langle A^2 \rangle_k$ shuffled to smaller scale.

$E_M < E_K \Rightarrow \langle A^2 \rangle_k$ transferred to larger scale.

and

transfer need not be local!

⇒ In dynamics, $\langle A^2\rangle$; $\langle A\cdot B\rangle$
evolution is complex.

⇒ N.B. Recall flux expulsion:

$\frac{V_{A0}^2}{\tilde{V}^2} R_m < 1 \rightarrow$ A ≈ passive, B expelled

$> 1 \rightarrow$ $J \times B$ disrupts vortex, expulsion stops

$$\Rightarrow \quad B_0^2 \lessgtr \rho_0 \langle \tilde{V}^2 \rangle / R_m$$

but $\langle \tilde{B}^2 \rangle \gg B_0^2$, upon stretching, weak B_0 is sufficient!

Zeldovich:

$$\frac{\partial A}{\partial t} + \underline{v}\cdot\underline{\nabla} A = -v_r \frac{\partial \langle A \rangle}{\partial x} + \eta \nabla^2 A$$

$\times A$ and avg. ⇒

$$\eta \langle \tilde{B}^2 \rangle = \langle \tilde{v}_r \tilde{A} \rangle \frac{\partial \langle A \rangle}{\partial x}$$

$$\langle \tilde{B}^2 \rangle = \frac{\eta_T}{\eta} B_0^2$$

$$= \frac{\tilde{v}\, \tilde{v} \tau_c}{\eta} B_0^2 \cong R_m B_0^2 \quad ✓.$$

so, crudely:

$$\langle \tilde{B}^3 \rangle / R_m \lessgtr \langle \tilde{v}^2 \rangle / R_m$$ ✓

$\Rightarrow$ Questions still open!

$\therefore$ Taylor conjecture remains a conjecture!